

SURFACES OF BOUNDED MEAN CURVATURE IN RIEMANNIAN MANIFOLDS

SIDDHARTHA GADGIL AND HARISH SESHADRI

ABSTRACT. Consider a sequence of closed, orientable surfaces of fixed genus g in a Riemannian manifold M with uniform upper bounds on mean curvature and area. We show that on passing to a subsequence and choosing appropriate parametrisations, the inclusion maps converge in C^0 to a map from a surface of genus g to M .

We also show that, on passing to a further subsequence, the distance functions corresponding to pullback metrics converge to a pseudo-metric of fractal dimension two.

As a corollary, we obtain a purely geometric result. Namely, we show that bounds on the mean curvature, area and genus of a surface $F \subset M$ together with bounds on the geometry of M give an upper bound on the diameter of F .

Our proof is modelled on Gromov's compactness theorem for J -holomorphic curves.

1. INTRODUCTION

The study of families of minimal surfaces and, more generally, constant mean curvature surfaces in Riemannian manifolds is a classical topic in differential geometry. Minimal surfaces in 3-manifolds, in particular, has received a lot of attention. In this paper, we focus our attention on closed (compact without boundary) surfaces in closed Riemannian n -manifolds with *bounded mean curvature*, generalising the case of surfaces of minimal surfaces (which are characterised by having mean curvature zero).

We prove a compactness result in a general setting. Let (M, g) be a closed, Riemannian n -manifold and let $H_0 \geq 0$ and $A_0 > 0$ be fixed constants. Consider a sequence of closed, connected orientable surfaces F_j in M of a fixed genus m with

- (1) the norm of the mean curvature vector field bounded above by H_0 and
- (2) area bounded above by A_0 .

Let $i_j : F_j \rightarrow M$ denote the inclusion maps. Let F be a smooth surface of genus m . Our main result says that after choosing appropriate parametrisations a subsequence of the surfaces converges in a C^0 -sense to a limiting (not necessarily embedded) surface.

Theorem 1.1. *There are homeomorphisms $\varphi_j : F_j \rightarrow F$ such that, after passing to a subsequence, the maps $i_j \circ \varphi_j^{-1}$ converge in the C^0 topology to a map $i_j : F \rightarrow M$.*

Date: November 13, 2008.

1991 *Mathematics Subject Classification.* Primary 57N10; Secondary 53A10.

UGC support. The second author was supported by DST Grant No. SR/S4/MS-283/05.

On passing to a further subsequence, we show that the distance functions corresponding to the pullback metrics converge to a pseudo-metric. We show further that the limit is in an appropriate sense 2-dimensional.

Theorem 1.2. *On passing to a subsequence, the distance functions d_j on F corresponding to the pullback metrics $g_j = (i_j \circ \varphi_j^{-1})^*g$ converge uniformly to a (continuous) pseudo-metric d on F . Moreover d has fractal dimension 2.*

As an application of Theorems 1.1 and 1.2, we prove the following purely geometric result.

Theorem 1.3. *Given a Riemannian manifold (M, g) and constants $A_0 > 0$ and $H_0 \geq 0$ and an integer $m > 0$, there is a constant $D = D(M, g, A_0, H_0, m)$, depending only on A_0, H_0, m and (M, g) , so that any surface $F \subset M$ of genus m with the norm of the mean curvature vector bounded above by H_0 and area at most A_0 has diameter at most D .*

Proof. We proceed by contradiction. Suppose $F_j \subset M$ is a sequence of surfaces satisfying the hypothesis with $\text{diam}(F_j) \rightarrow \infty$. By Theorem 1.1, on passing to a subsequence (which we also denote F_j) we can construct a family of diffeomorphisms $\varphi_j : F_j \rightarrow F$ to a fixed surface so that the associated maps $F \rightarrow M$ converge. Let $d_j : F \times F \rightarrow \mathbb{R}$ be the corresponding distance functions on F . Then $\text{diam}(F_j) = \sup_{(p,q) \in F \times F} d_j(p, q) \rightarrow \infty$.

By the first statement of Theorem 1.2, on passing to a further subsequence, the functions $d_j : F \times F \rightarrow \mathbb{R}$ converge to a continuous function $d : F \times F \rightarrow \mathbb{R}$, which is bounded as $F \times F$ is compact. It follows that the functions d_j are uniformly bounded above by the maximum of d , a contradiction. \square

There is a large body of literature dealing with families of minimal surfaces in Riemannian 3-manifolds. The foundational work of W. Allard [1] deals with weak convergence of minimal surfaces in n -manifolds. For surfaces in 3-manifolds, M. T. Anderson proved the following [2]: Let \mathcal{M}_n denote the space of minimal embeddings of a closed surface of genus $\geq n$ in a complete 3-manifold, endowed with the weak topology as a subset of the space of 2-varifolds. Then the boundary $\partial\mathcal{M}_n$ is contained in $\mathcal{M}_{\frac{n}{2}}$. As a corollary, it is shown that if the 3-manifold is compact and has negative sectional curvature, then \mathcal{M}_2 is compact.

For 3-manifolds of positive curvature, H. Choi and R. Schoen prove [3] the following compactness result: Let N be a closed 3-manifold of positive Ricci curvature. Then the space of closed embedded minimal surfaces of fixed topological type, endowed with the C^k topology, is compact if $k \geq 2$. In [11] B. White generalized the result of Choi and Schoen to stationary points of arbitrary elliptic functionals defined on the space of embeddings of a compact surface in a 3-manifold, minimal surfaces being stationary points of the area functional. White's result is that compactness holds for such surfaces if we assume a bound on the area.

More recently, W. Minicozzi and T. Colding have studied [4] sequences of minimal surfaces in 3-manifolds of bounded genus without bounds on area.

Outline of the proof of Theorem 1.1. Our compactness result and proof are modelled on Gromov's compactness theorem for J -holomorphic curves. However the ingredients in our situation - for instance the Schwarz lemma and the monotonicity lemma, need to be proved using different techniques than those for J -holomorphic curves. Furthermore, unlike the case of J -holomorphic curves (or

minimal surfaces), the surfaces we consider do not satisfy an elliptic partial differential equation, and hence we do not have regularity results.

Consider henceforth a sequence of surfaces F_i in the manifold M satisfying the above bounds on the mean curvature, area and genus. The surfaces F_i have Riemannian metrics obtained by restriction from M . We shall consider distances with respect to this metric. We shall frequently replace the given sequence by a subsequence, for which we continue to use the same notation.

The first elementary observation (see Section 2) is that the upper bound on mean curvature ensures a uniform upper bound on the sectional curvature of the surfaces.

The basic strategy of the proof is to puncture the surfaces about an ϵ -net and on the complement, to consider the complete hyperbolic metric in the conformal class of ι^*g . Here $\epsilon > 0$ is a sufficiently small constant depending only on the geometry of the ambient manifold M . A monotonicity lemma we prove in Section 3 shows that there is a uniform bound on the size of the ϵ -net. Hence by passing to a subsequence we may assume that the topological type of the punctured surfaces is fixed.

The Bers-Mumford compactness theorem says that, on passing to a subsequence, the hyperbolic surfaces have a limit which is a complete hyperbolic surface. The limiting surfaces in general has additional cusps created by pinching curves. Our goal is to construct a corresponding limit of maps. This depends on relating the hyperbolic metric on the surfaces to the metric obtained from M .

Specifically, we show that the away from the cusps the identity map from the surface with the hyperbolic metric to itself with the metric restricted from M is uniformly continuous. The first step (Section 4) is an argument that says that we have a form of uniform continuity at one scale. This uses an extremal length argument and the fact that an ϵ -net has been deleted. We then need an appropriate Schwarz lemma to conclude uniform continuity at all stages.

The version of Schwarz lemma we prove (in Section 5) and use is for discs with small diameter with an upper bound on the sectional curvature given. However this cannot be applied directly as it requires a lower bound on the injectivity radius at the origin (of the pullback metric). We apply the Schwarz lemma indirectly by constructing a lift of an appropriate disc under the exponential map. Such a lift is obtained (in Section 12) by a geometric argument, making crucial use of an upper bound on the perimeter of the disc.

Following Gromov's proof of the compactness theorem for J -holomorphic curves, the above ingredients allow us to construct a map on the punctured surface (see Section 6). Finally, further arguments using the extremal length, as well as a slightly more intricate one using in addition the monotonicity lemma, allow us to show that limits can also be obtained near the punctures in Section 7.

A word on notations. We shall be considering various surfaces called F_i , Σ_i , Σ , $\bar{\Sigma}$ and F . We clarify here what these mean (they will also be defined in appropriate places in the text).

The surfaces F_i are the given surfaces of bounded mean curvature, taken with their pullback metrics. The inclusion map from the surface F_j into M will be denoted i_j . We shall frequently pass to subsequences without changing our notation.

We shall construct a surface F , which is topologically of the same type as F_i with a pseudo-metric that is as a limit of the pullback metrics. This will be the domain of the limiting map $i : F \rightarrow M$.

The surfaces Σ_i are obtained from F_i by deleting a finite set of points, with the metric on Σ_i being the unique *hyperbolic* metric that is conformally equivalent to the pullback metric on F_i . The natural inclusion maps of Σ_j into F_j and M will be denoted ι_j and $\hat{\iota}_j$. We shall construct a hyperbolic surface Σ as a limit of the surfaces Σ_i . Finally, we shall compactify Σ and make certain identifications at infinity to obtain a surface $\bar{\Sigma}$.

CONTENTS

1. Introduction	1
2. Preliminaries	4
3. A Monotonicity Lemma and ϵ -nets	6
4. Hyperbolic structures	8
5. A Schwarz lemma	9
6. Limits of punctured surfaces	10
7. Filling punctures	11
8. Proof of Theorem 1.1	14
9. A limiting metric	15
10. Fractal dimension of the limit	16
11. Appendix A: Bounds on curvature and conjugate radius	21
12. Appendix B: Lifting discs under the exponential map	23
References	26

2. PRELIMINARIES

We assume throughout that all manifolds (and surfaces) we consider are orientable. Let M be a closed, smooth, Riemannian n -manifold which we fix throughout. For a fixed real number $H_0 \geq 0$, we consider embedded surfaces $F \subset M$ with mean curvature bounded above in absolute value by H_0 . In case $H_0 = 0$, these are just minimal surfaces. We further restrict to surfaces with area bounded above by A_0 and with a fixed genus g . The proofs of Lemma 2.1 and Lemma 2.2 are given in Appendix A.

2.1. Upper bounds on curvature. We begin by observing that there is an upper bound on the sectional curvature of the surfaces. This is used for a version of the Schwarz lemma.

Lemma 2.1. *There is a constant K_0 so that the sectional curvature of each surface F_i is bounded above by K_0 .*

2.2. Lower bound on conjugate radius. Let (F, g) be a Riemannian manifold. The *conjugate radius* at $p \in F$ is the largest R such that \exp_p is an immersion on $B(0, R) \subset T_p(M)$.

Lemma 2.2. *Let (F, g) be a complete Riemannian 2-manifold with sectional curvature bounded above by K_0 . Then the conjugate radius at any $p \in F$ is at least $R = \frac{\pi}{3\sqrt{K_0}}$. Moreover, if we write*

$$\exp^*(g) = dr^2 + f^2(r, \theta)d\theta^2$$

for polar coordinates (r, θ) on $T_p F$ and $r < R$, then $f(r, \theta)$ increasing as a function of r and $f(r, \theta) > r/2$ for all θ .

2.3. Lifting discs under the exponential map. We need to use the geodesic coordinates of Lemma 2.2 for a topological disc $D \subset F$ in a surface with an upper bound on the sectional curvature. However, the injectivity radius may be less than the diameter of D . We shall see, however, that we can lift discs with small diameter and small boundary under the exponential map. Let $R = R(K_0)$ be the constant from Lemma 2.2.

Lemma 2.3. *Let $\iota : B \rightarrow (F, g)$ be an immersion of a disc into a complete Riemannian 2-manifold (F, g) with sectional curvature bounded above by K_0 . Suppose that for the pullback metric ι^*g , the length of $\gamma = \partial B$ and the distance of a point in B to γ are both bounded above by $\epsilon < R/10$ where $R = \frac{\pi}{3\sqrt{K_0}}$. Then for $x = \iota(y)$ in the image of B , there is a lift $\tilde{\iota}$ of ι to the tangent space $T_x F$ so that $\iota = \exp_x \circ \tilde{\iota}$. Furthermore, the lift can be chosen so that $\tilde{\iota}(y)$ is the origin.*

We remark that ι (at least restricted to the interior of the disc B) is often the inclusion map on a subset of F . hence we identify p with x .

We prove this in Section 12

2.4. Conformal moduli of annuli. We recall some basic results regarding the conformal moduli of annuli that will be used extensively. An *annulus* is a 2-manifold homeomorphic to the product of a circle and an interval. We consider annuli with a given conformal class of Riemannian metrics (i.e., a conformal structure). Recall that this is equivalent to specifying a complex structure.

A *right circular annulus* $A(H, W)$ is the Riemannian product of a circle of circumference W and an open interval of (possibly infinite) length H . The following is the uniformisation theorem for annuli.

Theorem 2.4. *We have the following.*

- (1) *Any annulus A with a conformal structure is conformally equivalent to a right circular annulus.*
- (2) *$A(H_1, W_1)$ is conformally equivalent to $A(H_2, W_2)$ if and only if $H_1/W_1 = H_2/W_2$.*

From the above result, it is immediate that the following definition gives a well-defined number in $(0, \infty]$.

Definition 2.5. The modulus $Mod(A)$ of an annulus A with a conformal structure is $Mod(A) = H/W$ where A is conformally equivalent to the right circular annulus $A(H, W)$.

By definition the modulus is a conformal invariant. Further, two annuli are conformally equivalent if and only if they have the same modulus.

Observe that the area of a right circular annulus $A = A(H, W)$ is HW , so the modulus of $A(H, W)$ can also be expressed as $Mod(A) = Area(A)/W^2$, i.e., $W^2 = Area(A)/Mod(A)$. The following fundamental (though elementary) result of Ahlfors allows one to get an upper bound on the appropriate width for an annulus.

Theorem 2.6 (Ahlfors). *Let A be an annulus with a conformal structure. Then there is a simple closed curve $\gamma \subset A$ separating the two boundary components of A whose length $l(\gamma)$ satisfies*

$$l(\gamma)^2 \leq Area(A)/Mod(A)$$

Furthermore, given an identification of the annulus with $S^1 \times J$ for an interval J , we can find a curve γ as above of the form $S^1 \times \{p\}$.

It is easy to see that an annulus obtained from a disc by puncturing a point has infinite modulus. Further, if we take an open disc $D(r)$ of a fixed radius r in Euclidean or hyperbolic space and $D(\rho)$ is the concentric disc of radius $\rho < r$, then the modulus of the annulus $A = D(r) - \overline{D(\rho)}$ goes to infinity as $\rho \rightarrow 0$.

2.5. Real-analytic metrics and the cut-locus. Let (F, g) be a Riemannian manifold and $p \in F$. Let

$$U_p := \{v \in T_p M \mid \exp_p(tv) \text{ is a minimal geodesic on } [0, 1]\}.$$

Note that the boundary ∂U_p is the cut-locus in $T_p M$.

In case (F, g) is real-analytic we have the following proposition which follows from results of S. B. Myers.

Theorem 2.7. (*S. B. Myers [9], [10]*) *Let (F, g) be a closed real-analytic Riemannian 2-manifold and $p \in F$. Then*

(i) ∂U_p is a piecewise-smooth 1-manifold homeomorphic to S^1 .

(ii) Suppose that the sectional curvature of (F, g) is bounded above by K_0 . Let $R = \frac{\pi}{3\sqrt{K_0}}$ be the lower bound on conjugate radius given by Lemma 2.2. If $\delta < R$, then the geodesic sphere $\partial B(p, \delta)$ is a disjoint union of piecewise-smooth circles.

3. A MONOTONICITY LEMMA AND ϵ -NETS

As in the case of the monotonicity lemma for minimal surfaces, the proof of our monotonicity lemma is based on an isoperimetric inequality. The relevant isoperimetric inequality is due to Hoffman and Spruck. This will also be used later on in the paper.

Theorem 3.1. (Hoffman-Spruck [6]) *Let Σ be a compact surface with boundary in a Riemannian n -manifold (M, g) . There is a constant $v_0 = v_0(M, g)$ such that either $\text{vol}(\Sigma) \geq v_0$ or*

$$(1) \quad \text{vol}(\Sigma)^{\frac{1}{2}} \leq \beta \left(\text{vol}(\partial \Sigma) + \int_{\Sigma} |H| dV_{\Sigma} \right),$$

where β is an absolute constant.

Proof. This is a corollary of Theorem 2.2 of [6]. In the notation of that paper take $\alpha = \frac{1}{2}$ and $b = 1$, for instance. Let i_0 denote the injectivity radius of M . Then either (1) holds or

$$\text{vol}(\Sigma) \geq v_0 = \frac{\omega_2}{2} \min\left\{1, \frac{i_0^2}{\pi^2}\right\}.$$

□

The main result of this section is a monotonicity lemma, giving a lower bound on the area of small balls.

Theorem 3.2. *Let (M, g) a Riemannian n -manifold and Σ a compact surface in M with mean curvature $|H| \leq H_0$. There exist $c = c(M, g, H_0)$ and $\delta = \delta(M, g, H_0)$ such that the volume of any ball of radius $\epsilon \leq \delta$ in Σ with the induced metric satisfies*

$$\text{vol}(B(p, \epsilon)) \geq c\epsilon^2.$$

Proof. By making an arbitrarily small C^2 -perturbation we can assume that g is real-analytic. By Lemma 2.7 this implies that the boundary of a ball of radius ϵ less than R is piecewise-smooth. We apply Theorem 3.1 to the manifold with piecewise-smooth boundary $\partial B(p, \epsilon)$. Even (1) is stated for submanifolds with smooth boundaries, it is clearly true even if the boundaries are piecewise-smooth, as can be seen by exhausting such manifolds by submanifolds with smooth boundaries. We will apply it to the metric balls $B(p, r)$, $0 < r \leq \epsilon$.

If $\text{vol}(B(p, r)) \geq v_0$ for some $r < \epsilon$, then $\text{vol}(B(p, \epsilon)) > \text{vol}(B(p, r)) \geq v_0$. Hence $\text{vol}(B(p, \epsilon)) \geq \epsilon^2$ if $\epsilon < \sqrt{v_0}$.

So we can suppose that the isoperimetric inequality (1) holds for every $r \leq \epsilon$.

It follows from the co-area formula that $\text{vol}(B(p, r)) = \int_0^r \text{vol}(\partial B(p, t)) dt$. Hence $\text{vol}(B(p, r))$ is differentiable a.e. as a function of r .

By Theorem 3.1, we then have

$$\begin{aligned}
 \frac{d}{dr} \text{vol}(B(p, r)) &= \text{vol}(\partial B(p, r)) \\
 &\geq \beta^{-1} \text{vol}(B(p, r))^{\frac{1}{2}} - \int_{B(p, r)} |H| dV_\Sigma \\
 &\geq \beta^{-1} \text{vol}(B(p, r))^{\frac{1}{2}} - H_0 \text{vol}(B(p, r)) \quad \text{a.e.}
 \end{aligned}
 \tag{2}$$

We next see that we can assume $\beta^{-1} \text{vol}(B(p, r))^{\frac{1}{2}} > 2H_0 \text{vol}(B(p, r))$ for $0 \leq r \leq \epsilon$. If not, then we get $\text{vol}(B(p, \epsilon)) \geq \text{vol}(B(p, r)) \geq (\frac{1}{2\beta H_0})^2$, which is larger than ϵ^2 for $\epsilon < 2\beta H_0$.

As $\beta^{-1} \text{vol}(B(p, r))^{\frac{1}{2}} > 2H_0 \text{vol}(B(p, r))$, we get

$$\frac{d}{dr} \text{vol}(B(p, r)) \geq \frac{1}{2\beta} \text{vol}(B(p, r))^{\frac{1}{2}}$$

By integrating, we get $\text{vol}(B(p, \epsilon)) > \beta' \epsilon^2$ with $\beta' = \frac{1}{16\beta^2}$.

Hence we can take

$$\delta = \min\{R, \sqrt{v_0}, 2\beta H_0\}, \quad c = \min\{1, \frac{1}{16\beta^2}\}.$$

□

Gromov's proof of the compactness of J -holomorphic curves is based on puncturing along an ϵ -net. We shall choose an appropriate constant $\epsilon = \epsilon(M, H_0, A_0, g_0)$, which is the same for all the surfaces S_i . Assume that such a constant has been chosen. For each surface F_i , we choose a maximal subset $S_i \subset F_i$ so that the distance between every pair of points in S_i is at least ϵ , i.e., S_i is an ϵ -net.

Lemma 3.3. *There is a constant N such that for all i , $|S_i| \leq N$.*

Proof. Fix a surface F_i in the sequence. By hypothesis, the open balls $B(x, \epsilon) \subset F_i$, $x \in S_i$ are disjoint. By the monotonicity lemma, there is a constant a such that each of the balls have area at least a . As the area of F_i is bounded above by A_0 , the cardinality of S_i is bounded above by A_0/a . □

4. HYPERBOLIC STRUCTURES

Consider the sequence of surfaces $\hat{F}_j = F_j - S_j$. As the cardinality of S_j and the genus of F_j are bounded above, by passing to a subsequence we can, and do, assume that the surfaces \hat{F}_j are of a fixed topological type. Further, by ensuring that the number of punctures is at least three, we can ensure that $\chi(\hat{F}_i) < 0$. By the uniformisation theorem, there is a unique complete hyperbolic metric on \hat{F}_j that is conformal to the given Riemannian metric. We view this as a hyperbolic surface Σ_j which is identified with a subset of F_j .

By the Bers-Mumford compactness theorem, on passing to a subsequence the surfaces Σ_j converge to a complete, finite volume, hyperbolic surface Σ . More concretely, we have a sequence of numbers $\delta_j \rightarrow 0$, compact sets $\kappa_j \subset \Sigma_j$ and $(1 + \delta_j)$ -bi-Lipschitz diffeomorphisms $\psi_j : \kappa_j \rightarrow \Theta_j \subset \Sigma$ so that the sets Θ_j form an exhaustion of Σ . Furthermore, by passing to smaller sets, we can ensure that the sets Θ_j are complements of horocyclic neighbourhoods of the cusps of Σ_j (with the intersection of the neighbourhoods of each cusp being empty).

We shall show that the maps $\iota_j \circ \psi_j^{-1}$ are equicontinuous on compact sets (in a sense made precise below), so that the Arzela-Ascoli theorem allows us to construct a limiting map $\iota : \Sigma \rightarrow M$. We shall henceforth implicitly identify subsets of Σ (contained in Θ_j) with subsets of κ_j using the maps ψ_j . Under these identifications, the maps ι_j can be regarded as maps on subsets of Σ .

Consider now a compact set $K \subset \Sigma$. The injectivity radius on K is bounded below by a constant $\alpha > 0$. For j large, as above we can identify K with subsets $K_j \subset \Sigma_j$ and the injectivity radius on these sets is also bounded below by α as the map ψ_j is $(1 + \delta_j)$ -bi-Lipschitz with δ_j small for j large.

Our first step in proving equicontinuity is an upper bound on the diameter in the pullback metric of small hyperbolic balls of a fixed size.

Lemma 4.1. *There is a constant $r = r(\epsilon, \alpha, A_0, H_0, m, M)$ such that for any point $x \in K_j$, $\iota_j(B(x, r))$ is contained in a smooth (not in general metric) ball $B(\gamma)$ in $\widehat{F}_j \subset F_j$ of diameter 3ϵ whose boundary has length at most ϵ .*

Proof. We shall choose $r < \alpha$ appropriately. For a point $x \in K_j$, consider the annulus $A = B(x, \alpha) \setminus \text{int}(B(x, r))$. Choose r small enough that this annulus has modulus at least A_0/ϵ^2 . Note that this depends only on α , A_0 and ϵ .

Consider the annulus $\iota_j(A) \subset F_j$. This has area bounded above by A_0 . By Theorem 2.6, there is a curve γ in $\iota_j(A)$, separating the boundary components of A , so that the length of γ is bounded above by ϵ . The curve γ is the boundary of a ball $B(\gamma)$ that contains $\iota_j(B(x, r))$. We shall show that $B(\gamma)$ has diameter at most 3ϵ .

As $B(\gamma)$ is a ball whose boundary is a connected set of diameter at most ϵ , it suffices to show that for each point $x \in B(\gamma)$, the distance of x from the boundary is at most ϵ . To see this, observe that $B(\gamma) \subset F_j - S_j$ by construction. As S_j is an ϵ -net, the distance from $x \in B(\gamma)$ to some point $y \in S_j$ is at most ϵ . As the metric on F_j is obtained from a Riemannian metric and F_j is compact, there is a path β from x to y of length at most ϵ . This path must intersect the boundary of $B(\gamma)$ at some point z . It follows that $d(x, z) < \epsilon$. \square

5. A SCHWARZ LEMMA

To deduce equicontinuity of the maps $\iota_j : \Sigma_j \rightarrow \hat{F}_j$ from the estimate on diameters in the induced metrics of balls of radius ρ , we use an appropriate Schwarz lemma.

Theorem 5.1. *Let (S, g) be a Riemannian 2-manifold and $p \in \Sigma$ such that the sectional curvature of $S \leq K_0$ and injectivity radius $\text{inj}(p) \geq i_0$. Let (\mathbb{B}^2, h) denote the unit disc with the Poincaré metric of curvature -1 .*

For any $r > 0$, there exists $\eta = \eta(K_0, i_0)$ such that for any conformal map

$$f : B_h(0, r) \subset \mathbb{B}^2 \rightarrow B_g(p, \eta) \subset S,$$

with $f(0) = p$, we have

$$\|df_0(v)\|_g \leq r^{-2} \|v\|_h,$$

for all $v \in T_0 \mathbb{B}^2$.

Proof. The idea is to conformally deform g on $B_g(p, \eta)$ to $\tilde{g} = \exp(2u)g$ so that \tilde{g} has curvature ≤ -1 . The deformation will be done so that we have control over the conformal factor. Then we can apply the Ahlfors-Schwarz lemma in the new metric to get the required estimate.

First choose $\eta < \text{inj}(p)$. Define $u : B_g(p, \delta) \rightarrow \mathbb{R}^+$ by

$$u(x) = \lambda^2 \rho^2(x) \quad \text{and} \quad \lambda = \sqrt{\frac{K_0}{2} + 1},$$

where $\rho(x)$ is the distance of x from p . Since $\delta < \text{inj}(p)$, u is smooth. Now the curvature \tilde{K} of $\tilde{g} = \exp(2u)g$ is related to the curvature K of g by

$$\tilde{K} = (K - \Delta u) \exp(-2u),$$

where Δ is the negative Laplacian.

Since $K \leq K_0$, if $\rho(x) < \min\{\text{inj}(p), \frac{\pi}{4\sqrt{K_0}}\}$, we can apply the usual comparison theorem for the Laplacian of a distance function. Here the comparison space S_0 is the sphere of curvature K_0 . Let p_0, x_0 denote points in S_0 corresponding to p, x and let ρ_0 denote the distance from p_0 . We have $\rho_0(x_0) = \rho(x)$. The comparison theorem gives

$$(\Delta \rho)(x) \geq (\Delta_0 \rho_0)(x_0) = \cot(\sqrt{K_0} \rho(x)) = \frac{\cos(\sqrt{K_0} \rho(x))}{\sin(\sqrt{K_0} \rho(x))}.$$

In particular,

$$(\Delta \rho)(x) \geq 0,$$

if $\rho(x) < \min\{\text{inj}(p), \frac{\pi}{4\sqrt{K_0}}\}$. For such x , we have

$$\begin{aligned} (\Delta u)(x) &= \lambda^2 (\Delta \rho^2)(x) \\ &= 2\lambda^2 \rho(x) (\Delta \rho)(x) + 2\lambda^2 \|d\rho\|^2(x) \\ &\geq 2\lambda^2. \end{aligned}$$

(3)

We then have

$$\tilde{K}(x) \leq (K_0 - 2\lambda^2) \exp(-2\rho(x)^2) = -2 \exp(-2\rho(x)^2) \leq -1,$$

for $\rho(x) < \sqrt{\frac{\log(2)}{2}}$. Hence we can take

$$\eta = \frac{1}{2} \min \left\{ i_0, \frac{\pi}{4\sqrt{K_0}}, \frac{\log(2)}{2} \right\}.$$

Now we modify the metric on $B_h(0, r)$. Note that if $\phi(z) = r^2 \frac{(1-|z|^2)^2}{(r^2-|z|^2)^2}$, then $(B_h(0, r), \phi^2 h)$ is a complete Riemannian manifold of constant curvature -1.

Let us recall the

Ahlfors-Schwarz Lemma: Let (F_1, g_1) be a complete Riemannian 2-manifold with curvature $\geq -c$ and (F_2, g_2) a Riemannian 2-manifold with curvature $\leq -d$, where $c, d > 0$. If $f : (F_1, g_1) \rightarrow (F_2, g_2)$ is a conformal map, then

$$\|df_x(v)\|_{g_2} \leq \sqrt{\frac{c}{d}} \|v\|_{g_1},$$

for all $x \in F_1, v \in T_x F_1$.

We apply the Ahlfors-Schwarz Lemma to $f : (B_h(0, r), \phi^2 h) \rightarrow (B_g(p, \eta), \exp(2u)g)$ at $x = 0$. Noting that $c = d = 1$, $\phi(0) = r^{-2}$ and $\exp(2u(p)) = 1$, we get

$$\|df_0(v)\|_g \leq r^{-2} \|v\|_h.$$

□

6. LIMITS OF PUNCTURED SURFACES

The Schwarz Lemma of Theorem 5.1 requires a lower bound on the injectivity of the pullback metric, which we cannot control. However the following consequence of Lemma 2.3 allows us to obtain uniform Lipschitz bounds.

Note that the surface Σ can be identified with the complement of a collection of annuli in Σ_j for each j . Furthermore, if $K \subset \Sigma$ is a compact set, then for j large the corresponding set $K_j \subset \Sigma_j$ is close to an isometry.

Let $K \subset \Sigma$ be a compact set. Then there is a constant $\alpha > 0$ so that for j large, the injectivity radius of the set $K_j \subset \Sigma_j$ corresponding to K is bounded below by α . Let $\iota_j : K_j \rightarrow F_j$ be the inclusion map.

Lemma 6.1. *There is a constant κ , independent of j , so that the map ι_j is κ -Lipschitz for j sufficiently large.*

Proof. By Lemma 4.1, there is a constant $r > 0$ such that for any $p \in K_j$, $\iota_j(B(p, r))$ is contained in a topological ball $B(\gamma)$ whose diameter is at most 3ϵ and whose boundary has length at most ϵ in the pullback metric. Here $B(p, r)$ is the ball in the hyperbolic metric on Σ_j ,

By Lemma 2.3, there is a lift $\tilde{\iota}_j$ of ι_j to $T_p F_j$ under the exponential map \exp_p . Choose an isometry $\phi : B_h(0, r) \rightarrow B(p, r)$, with $\phi(0) = p$, where $B_h(0, r)$ denotes a ball in the Poincaré disc as in Theorem 5.1.

We can now apply the Schwarz Lemma of Theorem 5.1 to

$$f = \tilde{\iota}_j \circ \phi : B_h(0, r) \rightarrow B(0, R) \subset T_p F,$$

where $B(0, R)$ is endowed with the metric $\exp_p^* g$.

Since $\iota_j = \exp_p \circ \tilde{\iota}_j$ and $\exp_p : (B(0, R), \exp_p^* g) \rightarrow (F_j, g)$ is an isometry, we have

$$\|d\iota_j\|_{\phi(x)} = \|d\tilde{\iota}_j\|_{\phi(x)} = \|df\|_x.$$

Hence we obtain an upper bound on $\|d\iota_j\|$ at p for the map $\iota_j : K_j \rightarrow \hat{F}_j \hookrightarrow M$. Note that this upper bound does not depend on j , but depends only on M and α . This gives the uniform Lipchitz bound. \square

Lemma 6.2. *On passing to a subsequence, the maps $\hat{\iota}_j = \iota_j \circ \iota_j : \Sigma_j \rightarrow \hat{F}_j \rightarrow M$ converge uniformly on compact sets to a map $\iota : \Sigma \rightarrow M$.*

Proof. Let $K \subset \Sigma$ be a compact set. We apply Lemma 6.1 to the restrictions of the maps ι_j , regarded as maps on K , to obtain a uniform Lipshitz bound. Thus, by the Arzela-Ascoli theorem, there is a subsequence of surfaces so that the maps ι_j converge to a map $\iota : K \rightarrow M$. Now consider an exhaustion of Σ by compact sets $K^{(i)}$. By the above, we can find a subsequence of surfaces to obtain a limit on $K^{(1)}$. On passing to a further subsequence, we obtain a limit on $K^{(2)}$. Iterating this process and using a diagonal subsequence as usual, we obtain a limiting map on Σ . \square

7. FILLING PUNCTURES

We have constructed a limiting map from the punctured surface Σ of finite type to the manifold M . We show now that, on passing to a further sequence, we can construct a limiting map on a closed surface $\bar{\Sigma}$ obtained from Σ . This is the surface by filling in the punctures and making certain identifications of the filled in points.

Observe that there are two kinds of punctures (cusps). The first kind are those that correspond to the limits of punctures in $\Sigma_j = \hat{F}_j$ corresponding to points of the ϵ -net S_j . We denote the set of such punctures as $S(\Sigma)$. We can, and shall, identify these with points on the surfaces F_j .

The second kind are pairs of punctures formed in passing to the compactification of Moduli space by a sequence of curves $\alpha_j \subset \Sigma_j \subset F_j$ whose length in the hyperbolic metric on Σ_j goes to zero. We denote the set of such pairs of cusps by $\Lambda(\Sigma)$. Each point $p \in \Lambda(\Sigma)$ corresponds to a pair of ends p^\pm of Σ .

We consider the Freudenthal (end-point) compactification of Σ (where one point is added for each puncture) and identify points corresponding to pairs of punctures p^\pm , $p \in \Lambda(\Sigma)$. We denote the resulting surface by $\bar{\Sigma}$.

Thus,

$$\bar{\Sigma} = \Sigma \coprod S(\Sigma) \coprod \Lambda(\Sigma)$$

We shall extend the inclusion map to $\bar{\Sigma}$. As there are only finitely many punctures, it suffices to show that we can extend the map to the point in $\bar{\Sigma}$ corresponding to each puncture or pair of punctures.

We first consider a point $z \in S(\Sigma)$ corresponding to a limit of points of $z_j \in S_j$. As M is compact, by passing to a subsequence we can ensure that $\iota_j(z_j)$ converges to a point, which we take to be the image of z in the limiting map. It remains to show that this extension is continuous.

Suppose $D(z)$ is a closed disc in $\bar{\Sigma}$ containing the point z in its interior and no other points of $S(\Sigma)$ and $\Lambda(\Sigma)$. Then for j sufficiently large, the disc $D(z)$ can be identified with discs $D_j = D_j(z) \subset F_j = \Sigma_j$. Continuity is immediate from the following lemma.

Lemma 7.1. *Given $\delta > 0$, there is a disc $D(z)$ as above such that for j sufficiently large, the diameter of $\iota_j(D_j)$ is at most δ .*

We first give a brief sketch of the proof. Using an appropriate extremal length argument, we enclose D_j in a disc whose boundary has small perimeter. As before, we lift this disc under the exponential map. We then use polar co-ordinates, for which we have obtained the appropriate estimates in 2.2. These allow us to deduce a bound on the diameter of the disc from the bound on the perimeter. We now turn to the details.

Proof of Lemma 7.1. As the modulus of a disc punctured at a point is infinite, we can choose $D(z)$ so that there is an annulus $A(z)$ enclosing the puncture corresponding to z so that the modulus of the annulus is at least M , where M is any specified number. The annulus $A(z)$ corresponds to an annulus A in F_j which, for j sufficiently large, has modulus greater than M . Fix such a j .

Choosing M sufficiently large and using an extremal length argument as in Lemma 4.1, we can find a curve γ in A enclosing z_j with length $L(\gamma)$ less than η in the pullback metric, with $\eta > 0$ to be specified. Let $B(\gamma)$ be the disc bounded by γ enclosing z_j . We shall show that if η is small enough (depending only on δ and M), then the disc D_j has diameter less than δ . This implies that the diameter of the image $\iota_j(F_j)$ is less than δ .

We choose $\eta < \min(\delta/2, \epsilon/2)$. As the length of the boundary γ of $B(\gamma)$ is bounded by $\delta/2$, it suffices to show that the distance from a point x_0 of $B(\gamma)$ to γ is at most $\delta/3$. Suppose this is not the case, find a point x_0 with distance from the boundary greater than $\delta/3$.

Consider the exponential map from the tangent space at x_0 . We have seen that this is an immersion on the ball of radius 10ϵ . Further, as before the diameter of the set D_j is bounded by 2ϵ . We choose a lift as in Lemma 2.3 so that the image of x_0 is the origin.

We now recall Lemma 2.2.

We have assumed that the distance between x_0 and γ is greater than $\delta/3$. Hence the lift of γ (which we continue to denote by γ) encloses the ball of radius δ around the origin.

Consider the radial projection $p : \gamma \rightarrow \alpha$ of γ on to the boundary α of the ball of radius $\delta/3$. As $f(r, \theta)$ is an increasing function of r , this is distance decreasing. Thus, if ds_γ and ds_α denote the oriented arc lengths of the respective curves, $p^*(ds_\alpha) = \psi \cdot ds_\gamma$, with $\psi \leq 1$.

As γ encloses the origin, the projection has degree one (after possibly reversing the orientation of γ). Thus, we have

$$l(\alpha) = \int_\alpha ds_\alpha = \int_\gamma p^*(ds_\alpha) = \int_\gamma \psi \cdot ds_\gamma \leq \int_\gamma ds_\gamma = l(\gamma)$$

Hence $l(\alpha) \leq l(\gamma) < \eta$. Now by Lemma 2.2, it follows that $l(\alpha) > \pi\delta/3$. Hence, as $\eta < \delta/2 < \pi\delta/3$ we get a contradiction. \square

We now turn to the case of a point $p \in \Lambda(\Sigma)$, which corresponds to a pair of ends p^\pm . As before, we can find disc neighbourhoods of these, $D^\pm(p)$, enclosed by annuli $A^\pm(p)$ of modulus at least μ (see figure ??), with μ to be specified. We denote the boundaries of the disc by $\xi^\pm = \partial D^\pm$.

Consider now a surface Σ_j with j large. The curves ξ_j^\pm in F_j corresponding to ξ^\pm enclose an annulus B_j of large modulus (for j large) and hence are separated by a curve $\alpha_j = \alpha_j(p)$ of length l_j (in the pullback metric on F_j) so that as $j \rightarrow \infty$, $l_j \rightarrow 0$. Hence we can pass to a subsequence so that the images $\iota_j(\alpha_j)$ converge to

a point y in M . We extend the map so that the point p maps to y . We show that this extension is continuous.

Lemma 7.2. *Given $\delta > 0$, there are discs $D^\pm = D^\pm(p)$ as above so that for j sufficiently large the diameter of $B_j = B_j(p)$ is at most δ .*

Note that in this case, the upper bounds on the curvature and the length of the boundary do not suffice, as there are flat annuli with arbitrarily short boundaries with arbitrarily large diameters. We shall use an indirect argument, which we sketch below.

Consider an annulus as above whose boundary curves have length less than a small real number δ . As the annulus is contained in the complement of an ϵ -net, we can bound the distance between the two boundary components. This together with the bound on the boundary components and the upper bound on curvature gives an upper bound on the area of the annulus. By ensuring ϵ (hence δ) is sufficiently small, we can thus ensure that we are not in the *bubbling* case of the isoperimetric inequality (i.e., we ensure that volume is less than $v_0 = v_0(g)$ of Theorem 3.1). Hence, as the upper bound δ on the perimeter tends to zero, so does the area. Using the monotonicity lemma, we can deduce that the diameter also tends to 0. We now turn to the details.

Proof of Lemma 7.2. As the annuli A_j^\pm have modulus greater than μ , by choosing μ sufficiently large we can ensure that there are curves γ_j^\pm separating the boundary components of the annuli with lengths at most ϵ . We let $C_j = C_j(p)$ be the annulus enclosed by the curves γ_j^\pm . We shall show that the diameter of C_j is small, from which a bound on the diameter of the image of B_j follows.

Observe that as the annulus C_j is contained in the complement of an ϵ -net in F_j , as before the boundary components are at most a distance 2ϵ apart. Namely, as before each point is a distance less than ϵ from at least one of the boundary components. By connectedness, some point must have distance less than ϵ from both the boundary components.

Choose an embedded arc β of length at most 2ϵ joining the boundary components. We obtain a disc Δ_j from the annulus C_j by splitting along the curve β . Observe that all the above results continue to hold when the inclusion map of the disc is replaced by the quotient map from the disc to A_j . Choosing $\eta < \epsilon/2$, the disc Δ_j has boundary of length at most 6ϵ .

As before, we pick a point x_0 in B and find a lift of Δ_j with respect to the exponential map based at x_0 so that the lift of x_0 is the origin. Recall that there is a constant $v_0 = v_0(B(g)) > 0$ associated with the isoperimetric inequality (Theorem 3.1) which depends only on M .

Lemma 7.3. *If $\epsilon > 0$ is chosen sufficiently small, depending only on M , then the area of Δ_j with the pullback metric is less than v_0 .*

Proof. As the the origin is in the interior of Δ_j and the distance of any point in $B(\gamma)$ to the boundary is at most ϵ , the disc Δ_j is contained in the ball of radius ϵ around the origin.

Using polar co-ordinates as before, the area form on Δ_j is $\omega = f(r, \theta) dr \wedge d\theta$. Let $F(r, \theta) = \int_0^r f(\rho, \theta) d\rho$ and $\alpha = F(r, \theta) d\theta$. Then the area form $\omega = d\alpha$, hence

by Stokes theorem

$$Area = \int_{\Delta_j} \omega = \int_{\partial\Delta_j} \alpha$$

Let $\zeta = \partial\Delta_j$. In polar co-ordinates, we can express $\zeta = (\zeta_r, \zeta_\theta)$. In terms of these co-ordinates,

$$Area = \int_{\partial\Delta_j} \alpha = \int F(r, \theta) \frac{d\zeta_\theta}{dt} dt$$

Note that as $f(r, \theta)$ is increasing as a function of R , $F(r, \theta) \leq \epsilon f(r, \theta)$ for $r < \epsilon$. Further, by Pythagoras theorem the oriented arc length ds satisfies $ds > \frac{d\zeta_\theta}{dt}$. Thus, it follows that

$$Area = \int F(r, \theta) \frac{d\zeta_\theta}{dt} dt < \epsilon l(\zeta)$$

As $l(\zeta) < 5\epsilon$, the lemma follows. \square

Assume $\epsilon > 0$ has been chosen to satisfy the previous lemma. Then, by the isoperimetric inequality, if η is sufficiently small, the volume of C_j is less than $V(\eta)$, where $V(\eta)$ is a function of η such that $V(\eta) \rightarrow 0$ as $\eta \rightarrow 0$.

We show that if $\eta > 0$ is sufficiently small, then the diameter of the annulus C_j is less than δ . As before, it suffices to show that the distance of each point $x_0 \in C_j$ to the boundary of the annulus is at most $\delta/4$.

Suppose the distance of x_0 to ∂C_j is greater than $\delta/4$. It follows by the monotonicity lemma that the area of C_j is at least V_δ , with V_δ depending only on δ and M . Choose η such that $V(\eta) < V(\delta)$, to get a contradiction. Thus, we get a bound on the diameter showing continuity as required. This completes the proof of Lemma 7.2. \square

8. PROOF OF THEOREM 1.1

We can now complete the proof of Theorem 1.1. We have already constructed a limiting map on $\bar{\Sigma}$. We shall express $\bar{\Sigma}$ as the quotient of a surface F and construct appropriate diffeomorphisms $\varphi : F_j \rightarrow F$ so that the maps $i_j \circ \varphi^{-1}$ converge.

Assume that we have chosen a subsequence so that we have a corresponding limit map ι on $\bar{\Sigma}$. Let F be a surface of genus m . Identify $S(\Sigma)$ with a subset $S(F)$ of F . Then for a collection of disjoint curves $\alpha(p) \subset F - S(F)$, $p \in \Lambda(\Sigma)$, we have a homeomorphism of Σ with $F - \cup_{p \in \Lambda(\Sigma)} \alpha(p) - S(F)$. This extends continuously to a homeomorphism of the quotient of F with each curve $\alpha(p)$ identified to a point, which is mapped to $p \in \bar{\Sigma}$. Choose and fix a corresponding quotient map $q : F \rightarrow \bar{\Sigma}$. Let $i : F \rightarrow M$ be the composition $i = \iota \circ q$. Observe that Σ can be identified with a subset of F so that q the identity map on Σ and $i = \iota$ on Σ .

We construct next diffeomorphisms $\varphi_j : F_j \rightarrow F$. These will be extensions of the diffeomorphisms $\psi_j : \kappa_j \rightarrow \Theta_j \subset \Sigma$ of Section 4 using the identification of Σ_j with a subset of F_j .

We recall that the set Θ_j is the complement of a union of horocyclic neighbourhoods of cusps. Thus, there are punctured discs $\Delta(j; z) \subset \Sigma$, $z \in S(\Sigma)$ and $\Delta^\pm(j; p) \subset \Sigma$, $p \in \Lambda(\Sigma)$ so that Θ_j is the complement of the interiors of the sets

$D(j; z)$ and $D^\pm(j; p)$. Without loss of generality we assume that $k \geq j$ implies that $\Delta(k; z) \subset \Delta(j; z)$ and $\Delta^\pm(k; p) \subset \Delta^\pm(j; p)$ for all z and p .

We shall make use of the constructions of Lemmas 7.1 and 7.2. As the sets Θ_j form an exhaustion of Σ , for each fixed $z \in S(\Sigma)$ or $p \in \Lambda(\Sigma)$, the intersection of the corresponding punctured discs $\Delta(j; p)$ and $\Delta^\pm(j; p)$ is empty. It follows as in the proofs of Lemmas 7.1 and 7.2 that if $D_k(z)$ and $B_k(p)$ denote the corresponding discs and annuli in F_k for k sufficiently large, then the diameters of these sets are bounded above by δ_j so that $\delta_j \rightarrow 0$ as $k \rightarrow \infty$.

We take φ_j to be an extension of ψ_j so that the complement of the set $\kappa_j \subset F_j$ is mapped to the complement of the set $\Theta_j \subset \Sigma \subset F$. Any compact set $K \subset \Sigma$ is contained in Θ_j for j sufficiently large. Hence, $i_j \circ \varphi^{-1} = i_j \circ \psi_j^{-1}$ converges to $i = \iota : \Sigma \rightarrow M$. Finally, for the points of $S(F)$ and points on α_i , by the continuity of the extension of $\iota : \Sigma \rightarrow M$ to $\bar{\Sigma}$, we see that $i_j \circ \varphi^{-1} : F \rightarrow M$ converges to $i = \iota \circ q : F \rightarrow M$.

9. A LIMITING METRIC

We continue to use the notation of the previous section. Using the diffeomorphisms φ_j , we can identify the surfaces F_j with F . Hence the pullback metrics on the surface F_j give Riemannian metrics on F which have associated distance functions d_j . We shall construct a limiting pseudo-metric d on the surface F . Recall that this is a symmetric function $d : F \times F \rightarrow [0, \infty)$ that satisfies the triangle inequality but not in general the positivity condition for metrics.

Theorem 9.1. *On passing to a subsequence, the metrics d_j converge uniformly on F to a pseudo-metric d .*

Proof. Let g_0 denote the fixed hyperbolic metric on Σ and d_0 the distance function of this metric. Our first step is to show that the distance functions d_j converge on Σ .

Lemma 9.2. *The family of function $d_j : \Sigma \times \Sigma \rightarrow \mathbb{R}$ is equicontinuous on every compact subset K of $\Sigma \times \Sigma$, where we consider the product metric obtained from d_0 on $\Sigma \times \Sigma$.*

Proof. Showing equicontinuity on the product is equivalent to showing that given $\eta > 0$ there is a $\delta > 0$ so that for pairs (p_i, q_i) , $i = 1, 2$, such that $d_0(p_1, p_2) < \delta$ and $d_0(q_1, q_2) < \delta$, we have $|d_j(p_1, q_1) - d_j(p_2, q_2)| < \eta$. Note that by the triangle inequality,

$$|d_j(p_1, q_1) - d_j(p_2, q_2)| < d_j(p_1, p_2) + d_j(q_1, q_2)$$

Hence it suffices to show that we can choose δ so that $d_0(p_1, p_2) < \delta$ implies $d_j(p_1, p_2) < \eta$. But this follows from Lemma 6.1, where we have shown the uniform Lipschitz property of the functions d_j on compact subsets of Σ . \square

We shall apply the above to the compact sets $\Theta_j \in \Sigma = F$ of the previous section, which exhaust Σ . By the above Lemma the distance functions d_k are uniformly Lipschitz on Θ_j for each j . Hence, we can iteratively pass to subsequences and use a diagonal sequence to ensure that the metrics d_k have a limit d with convergence uniform on each set $\Theta_j \times \Theta_j$. More precisely, we can ensure that if $p, q \in \Theta_j$ and $k \geq j$, then $|d_k(p, q) - d(p, q)| < \delta'_j$ with $\delta'_j \rightarrow 0$ as $j \rightarrow \infty$.

As d is the limit of metrics, it is a pseudo-metric on Σ . Let $\bar{\Delta}(j; z)$ denote the closure of $\Delta(j; z)$ in F and $\bar{\Delta}(j; p)$ denote the closure of $\Delta^+(j; p) \cup \Delta^-(j; p)$ in F .

Observe that these sets are identified with sets $D_j(z)$ and $B_j(p)$ under the map φ_j and hence have diameter at most δ_j in the metric d_k for k sufficiently large (as in the previous section). Hence the diameter of the sets $\bar{\Delta}(j; z)$ and $\bar{\Delta}(j; p)$ in the pseudo-metric d is at most δ_j .

We shall extend d to F by continuity. Consider first the case where $p \in \Sigma$ and $q \in \alpha(p)$ for some $p \in \Lambda(\Sigma)$. Consider a sequence q_i in Σ converging to q . Without loss of generality we can assume that $q_j \in \bar{\Delta}(j; p)$ for some p . We claim that the sequence $d(p, q_j)$ is Cauchy. For $k \geq j$, as $q_k \in \bar{\Delta}(k; p) \subset \bar{\Delta}(j; p)$ and the diameter of $\bar{\Delta}(j; p)$ in the pseudo-metric d is at most δ_j , $d(q_j, q_k) \leq \delta_j$. Hence, by the triangle inequality

$$|d(p, q_j) - d(p, q_k)| \leq d(q_j, q_k) \leq \delta_j$$

It follows that the sequence $d(p, q_j)$ is Cauchy and hence converges to a limit which we define to be $d(p, q)$. Observe that if q'_j is another sequence in $\bar{\Delta}(j; p)$, then as above $|d(p, q_j) - d(p, q'_j)| < \delta_j$. Hence the limit is well-defined.

We can similarly define $d(p, q)$ if $q \in S(\Sigma)$. In case neither p nor q are in Σ , we use sequences p'_j and q'_j in Σ converging to these points. As above we get Cauchy sequences with limit independent of the choice of sequence.

We finally show that the convergence is uniform on all of F . Suppose now that $p, q \in F$ are arbitrary. We shall find a uniform upper bound for the quantity $|d_j(p, q) - d(p, q)|$. Suppose $p, q \in \Theta_j$, then by as above $|d_j(p, q) - d(p, q)| \leq \delta'_j$. Otherwise, one of p and q is in the interior of some set of the form $\bar{\Delta}(j; p)$ or $\bar{\Delta}(j; z)$. We consider the case when $p \in \Theta_j$ and $q \in \bar{\Delta}(j; p)$ as the other cases are similar.

Let q' be a point in the boundary $\partial\bar{\Delta}(j; p) = \bar{\Delta}(j; p) \cap \Theta_j$. Then as above, $|d_j(p, q') - d(p, q')| < \delta'_j$. Further, as the set $\bar{\Delta}(j; p)$ has diameter at most δ_j in the (pseudo)metrics d_j and d , $d(q, q') \leq \delta_j$ and $d_j(q, q') \leq \delta_j$.

By the triangle inequality, it follows that

$$|d_j(p, q) - d(p, q)| \leq 2\delta_j + \delta'_j$$

This shows that we have uniform convergence of the metrics d_k to d on all of F \square

We shall use the notation

$$|d_j - d| := \sup_{p, q \in F} |d_j(p, q) - d(p, q)|$$

10. FRACTAL DIMENSION OF THE LIMIT

We have constructed a limiting metric d on the surface F . In this section, we show that this metric has fractal dimension two and has finite, non-zero area in an appropriate sense. This gives a proof of Theorem 1.2

Now we come to the main lemma of this section.

Lemma 10.1. *Let (F, g) be a compact Riemannian 2-manifold with sectional curvature $K \leq K_0$. For any $p \in F$ and $0 < \delta < \frac{1}{\sqrt{3K_0}}$, if*

$$\text{vol}(B(p, \delta)) > C\delta^2,$$

then there is a $\delta' < \delta$ with

$$\int_{B(p, \delta')} K dv \leq 2\pi - \frac{C}{2}.$$

Proof. By making an arbitrarily small C^2 perturbation of g we can assume that it is real-analytic. Recall that $\frac{\pi}{3\sqrt{K_0}}$ is the uniform lower bound on the conjugate radius given by Corollary 2.2. Hence

$$\tilde{g} := \exp^*(g)$$

is a Riemannian metric on the (Euclidean) ball $B(0, \delta)$ in $T_p M$. Also,

$$\exp_p : \text{Int}(U \cap B(0, \delta)) \rightarrow \text{Int} B(p, \delta)$$

is a diffeomorphism, where $U = U_p$ as in Section 2.5. Note that U is star-shaped with respect to $0 \in T_p M$.

If we let $P = U \cap B(0, \delta)$, then by Lemma 2.7 ∂P is a piecewise-smooth 1-manifold. In fact we can write

$$\partial P = L_1 \cup \dots \cup L_k \cup S_1 \cup \dots \cup S_l,$$

where each S_i is a smooth closed segment contained in the circle $S(0, \delta)$ and each L_i is a smooth closed segment contained in the cut-locus ∂U . We refer to the L_i and S_i as *sides* of P and the non-smooth points of ∂P as *vertices* of P . We can assume that a side intersects another side in at most one point.

By changing δ slightly, we can assume that vertices of ∂P are either common points of a side L_i and a side S_j or two sides L_i and L_j .

By the definition of the cut locus, for each L_i there is at least one L_j , $j \neq i$, such that $\exp_p(L_i) = \exp_p(L_j)$. For any such pair, $L_i \cap L_j = \emptyset$, since we know that \exp_p is a local diffeomorphism on $B(0, \delta)$.

Lemma 10.2. *We have the following.*

- (1) $\sum_{i=1}^k \int_{L_i} \kappa = 0$
- (2) *The angle between any two consecutive sides of P is positive.*

Proof. As δ is less than the conjugacy radius, we have Gauss normal coordinates on P which we denote r and θ as usual, with the coordinates of a point x denoted $r(x)$ and $\theta(x)$.

Let L_j be a segment as above parametrised by a function $\alpha(s)$, with $\alpha(0)$ a point in the interior of L_j . Then, as L_j is in the boundary of P , there is a subsegment of some segment L_i , parametrised by $\beta(s)$, so that $\alpha(s)$ and $\beta(s)$ have the same image $c(s)$ under the exponential map and $d(\alpha(s), p) = d(\beta(s), p)$, i.e. $r(\alpha(s)) = r(\beta(s))$, in a neighbourhood of $s = 0$. The images of the radial vectors joining p to $\alpha(s)$ and $\beta(s)$ form geodesics γ_s and ξ_s of the same length joining p to $c(s)$.

By differentiating $r(\alpha(s)) = r(\beta(s))$ and considering images in F , we see that the inner products of $\alpha'(0)$ and $\beta'(0)$ with the respective radial vectors are equal. As $\alpha(s)$ and $\beta(s)$ have the same image on a neighbourhood of 0, $\alpha'(0)$ and $\beta'(0)$ have the same norm. It follows that the angles made by the vectors $\alpha'(0)$ and $\beta'(0)$ with the respective unit radial vectors are the same. On passing to the image, we see that the geodesics γ_0 and ξ_0 make the same angle with $c'(0)$ at the point $c(0)$.

It follows that γ_0 and ξ_0 must approach $c(0)$ on opposite sides – otherwise they would have a common point and direction and hence coincide. In particular, there are exactly two points in ∂P that map to a smooth (even C^1) point on the cut-locus, for if there were at least three points two would be on the same side. It follows that the segments L_j are identified in pairs, with the interior of P mapping to opposite sides of the image.

We deduce that

$$\sum_{i=1}^k \int_{L_i} \kappa = 0,$$

where κ denotes geodesic curvature. This is because the terms that correspond to L_i and L_j mapping to the same segment have equal magnitude (as their image is equal) and opposite signs (as the interior of P maps to opposite sides of the image).

We next see that the internal oriented angle θ_i between any two consecutive closed segments in $U \cap B(0, \delta)$ is positive. Observe that as P is a star convex region with boundary piecewise smooth, we can parametrise ∂P by angle using a function $r(\theta)$. This is smooth wherever ∂P is smooth. The left and right derivatives $r'_\pm(\theta)$ exist at all points. All internal oriented angles are positive if and only if for every non-smooth point (i.e., vertex) θ , $r'_-(\theta) \geq r'_+(\theta)$.

Consider first the case when a vertex of ∂P between an edge of the form L_i and one of the form S_j . By construction, on S_j we have $r(\theta) = \delta$ and on L_i we have $r(\theta) \leq \delta$. It is immediate that $r'_-(\theta) \geq r'_+(\theta)$.

Next, consider a vertex v between segments L_{i_1} and L_{i_2} , parametrised by $\alpha_1(s)$, $s \leq 0$ and $\alpha_2(s)$, $s \geq 0$, respectively. By the above, there are edges L_{j_1} and L_{j_2} that can be parametrised by curves β_1 and β_2 with the images of α_i and β_i coinciding and $d(p, \alpha_i(s)) = d(p, \beta_i(s))$.

We see that the curves α_i and β_i can be extended so that their domain of definition includes a neighbourhood of the origin and with the images of α_i and β_i coinciding and $d(p, \alpha_i(s)) = d(p, \beta_i(s))$. Without loss of generality, we prove this for $\alpha = \alpha_1$ and $\beta = \beta_1$. Namely, as δ is less than the conjugacy radius, the exponential map gives diffeomorphisms from neighbourhoods U_α and U_β of $\alpha(0)$ and $\beta(0)$ to a neighbourhood V of their image v . The images of the coordinate function r under these diffeomorphisms gives coordinate functions r_α and r_β . The condition $d(p, \alpha_i(s)) = d(p, \beta_i(s))$ is equivalent to $r_\alpha = r_\beta$.

The gradients of the functions r_α and r_β at v are unit vectors along the geodesic segments from p to the vertex v that are the images of the radial vectors to $\alpha(0)$ and $\beta(0)$. As these geodesics do not coincide, the gradients do not coincide and hence the gradient of $r_\alpha - r_\beta$ is non-zero. It follows that the set $r_\alpha = r_\beta$ is a manifold near v . Taking inverse images under the diffeomorphisms from U_α and U_β to V gives the required smooth extensions of α and β .

We now consider these extensions of α_1 and α_2 . By the definition of the cut-locus, it follows that for $s > 0$, $r(\alpha_2(s)) \leq r(\alpha_1(s))$. As $\alpha_1(s)$ is smooth at 0, we deduce that $r'_-(\theta) \geq r'_+(\theta)$ if $\theta = \alpha(0)$ is the given vertex. Thus, in this case too the angles are positive. This completes the proof of the lemma. \square

We now return to the proof of Lemma 10.1. Since $\delta < \frac{\pi}{3\sqrt{K_0}}$, we can compare the Riemannian manifold $(B(0, \delta), \tilde{g})$ with the round 2-sphere of radius $\frac{1}{\sqrt{K_0}}$ (the advantage of working with $B(0, \delta)$ in $T_p M$ rather than $B(p, \delta)$ in M is that the injectivity radius of $B(0, \delta)$ with the pull-back metric \tilde{g} is δ). If κ denotes the mean-curvature function on $S(0, \delta)$, then

$$\kappa \geq \sqrt{K_0} \frac{\cos(\sqrt{K_0}\delta)}{\sin(\sqrt{K_0}\delta)} \frac{1}{2\delta} \quad \text{on} \quad S(0, \delta).$$

Now we can apply the Gauss-Bonnet theorem to get

$$\begin{aligned}
 \int_{B(p,\delta)} K &= \int_{U \cap B(0,\delta)} \tilde{K} \\
 &= 2\pi\chi(B(0,\delta)) - \sum_{i=1}^k \int_{L_i} \kappa - \sum_{j=1}^l \int_{S_j} \kappa - \sum_{i=1}^k \theta_i \\
 &\leq 2\pi - \sum_{j=1}^l \int_{S_j} \kappa \\
 (4) \quad &\leq 2\pi - \frac{l(\delta)}{2\delta},
 \end{aligned}$$

where \tilde{K} is the Gaussian curvature of \tilde{g} and

$$l(\delta) = \sum_{j=1}^l \text{length}(S_j).$$

Note that we have used the Lemma 10.2 proved earlier in going from line 2 to line 3 above. The area of $B(p, \delta)$ is given by

$$\text{vol}(B(p, \delta)) = \text{vol}(U \cap B(0, \delta)) = \int_0^\delta l(s) ds,$$

Hence if $\text{vol}(B(p, \delta)) \geq C\delta^2$ for some C , then there exists $\delta' \leq \delta$ with

$$l(\delta') \geq C\delta.$$

By (4), we would have for this δ'

$$\int_{B(p,\delta')} K \leq 2\pi - \frac{l(\delta')}{2\delta'} \leq 2\pi - \frac{C}{2}.$$

This completes the proof of Lemma 10.1. □

Corollary 10.3. *Let (F, g) be a compact Riemannian 2-manifold with sectional curvature $K \leq K_0$ and area A satisfying $a_0 \leq A \leq A_0$. Suppose that $S = \{x_1, \dots, x_l\}$ is a δ -net with $\delta < \frac{1}{\sqrt{2K_0}}$.*

Then, for any $x_i \in S$, we have

$$\text{vol } B(x_i, \delta) \leq C_0 \delta^2,$$

where

$$C_0 = 8\pi^2(1 - \chi(F)) + 4\pi K_0 A_0.$$

Hence it follows that the cardinality of S is at least $\frac{a_0}{C_0} \delta^{-2}$.

Proof. We will assume that at some point, say x_1 ,

$$\text{vol } B(x_1, \delta) \geq C_0 \delta^2$$

and get a contradiction. The Gauss-Bonnet theorem applied to F along with Lemma 10.1 gives

$$\begin{aligned} 2\pi\chi(F) &= \int_{B(x_1, \delta')} K dv + \int_{F-B(x_1, \delta')} K dv \\ &\leq 2\pi - \frac{C_0}{2} + K_0 A_0. \end{aligned}$$

This gives

$$C_0 < 4\pi(1 - \chi(F)) + K_0 A_0,$$

a contradiction. \square

Let F be a surface in M with the given bounds on mean curvature, genus and area bounded above by A_0 . Then the sectional curvature of F is bounded above. We next see that there is a lower bound a_0 on the area of F depending only on the geometry of M and the given bounds on F . This allows us to apply the above corollary uniformly.

Lemma 10.4. *There is a constant a_0 depending only on the geometry of M and the bound on the mean curvature of F such that the area of F is at least a_0 .*

Proof. This follows from Theorem 3.2 applied for an arbitrary value of ϵ . \square

We next show that there is a lower bound on the area of a ball of radius δ , hence an upper bound on the size of a δ -net.

Lemma 10.5. *There is a constant $c > 0$ such that for δ sufficiently small, the area of the ball of radius δ in F around a point $p \in F$ is at least $c\delta^2$. As a consequence the size of a δ -net is at most $A_0/c(\delta/2)^2$.*

Proof. Again, by using Theorem 3.2, we can deduce the bound on the size of the δ -net. Note that for a δ -net S , the balls of radius $\delta/2$ centered around the points of S are disjoint. Hence their total area is at most the area of F , which is in turn at most A_0 . As the area of each of these balls is at least $c(\delta/2)^2$, it follows that the cardinality of S is at most $A_0/c(\delta/2)^2$. \square

We conclude that the size of a δ -net for the metric d grows as δ^{-2} as $\delta \rightarrow 0$.

Theorem 10.6. *There are constants $0 < b < B < \infty$ such that, for δ sufficiently small, the size of a δ -net S for the pseudometric d satisfies*

$$b\delta^{-2} \leq |S| \leq B\delta^{-2}$$

Proof. Suppose $\delta > 0$ is sufficiently small and S is a δ -net for d , i.e., a maximal set so that all pairwise distances are at least δ . Let j be such that $|d_j - d| < \delta/2$. Then for $p, q \in S$, $p \neq q$, we have $d_j(p, q) \geq \delta/2$. Hence S is contained in a $\delta/2$ -net S' for the metric d_j . But Lemma 10.5 gives an upper bound of the form $B\delta^{-2}$ for the cardinality of S' , and hence of S .

Next, let T be a 3δ -net for the metric d_j . We claim that the cardinality of T is at most that of S . First observe that as S is a δ -net, if $x \in F$ then for at least one $p = p(x)$ in S , $d(x, p) \leq \delta$. If $x \in S$, this is obvious, otherwise be considering $S \cup \{x\}$ we get a contradiction to the maximality. As $|d_j - d| < \delta/2$, it follows that $d_j(x, p(x)) < 3\delta/2$.

For each point $q \in T$, choose and fix $p(q)$ as above. This gives a function $p : T \rightarrow S$.

Lemma 10.7. $p : T \rightarrow S$ is injective.

Proof. Suppose $p(q) = p(q') = p$. Then we have seen that $d_j(q, p) \leq 3\delta/2$ and $d_j(q', p) < 3\delta/2$. By the triangle inequality, $d_j(q, q') < 3\delta$, contradicting the hypothesis that T is a 3δ -net for the metric d_j . \square

It is immediate that the cardinalities of S and T satisfy $|T| \leq |S|$. But Corollary 10.3 gives a lower bound of the form $b\delta^{-2}$ on $|T|$, hence on the cardinality of S . \square

A coarse notion of area (and volume), and the corresponding notion of dimension, the so called *fractal dimension*, can be defined in terms of δ -nets. Namely, let (X, d) is a metric space. For $\delta > 0$, let $n(\delta)$ be the minimum number of balls of radius δ that cover X . For $s \geq 0$, define the s -dimensional volume by

$$V^s(X) = \limsup_{\delta \rightarrow 0} n(\delta) \delta^s$$

It is an immediate consequence of Theorem 10.6 that the 2-dimensional volume, in the above sense, of F with the metric d is a finite, positive number. Further for $s < 2$ the s -dimensional volume is zero and for $s > 2$ it is infinite. Thus, the limiting metric on the surface is a metric of fractal dimension two and of finite, positive 2-dimensional volume.

The fractal dimension is closely related to, but not equal to, the Hausdorff dimension. In particular, it is a *capacity* rather than a measure - we have finite but not countable additivity. For example, if X is the set $\mathbb{Q} \cup [0, 1]$ of rational numbers in $[0, 1]$, then $V^1(X) = 1$ but the 1-dimensional Hausdorff measure of X is zero.

It is easy to deduce from the above that the Hausdorff dimension of F is at most 2. However, we do not know whether the Hausdorff dimension must be two. We remark that the metric d_j is not in general bilipshitz to the pseudometric d as for pairs of distinct points p, q in a circle α^i in F as above (if there is at least one such circle), $d(p, q) = 0$ but $d_j(p, q) \neq 0$.

11. APPENDIX A: BOUNDS ON CURVATURE AND CONJUGATE RADIUS

Lemma (see Lemma 2.1). *Let F be an embedded surface in a Riemannian n -manifold (M, g) with mean curvature bounded above by H_0 . There is a constant $K_0 = K_0(M, g, H_0)$ so that the sectional curvature of F is bounded above by K_0 .*

Proof. Let $\tilde{\nabla}$ and ∇ denote the Riemannian connections of M and F . Fix $p \in F$ and let N_1, \dots, N_{n-2} be unit normal vector fields defined in a neighbourhood of p . Then the second fundamental form B is given by

$$B(X, Y) := \tilde{\nabla}_X Y - \nabla_X Y$$

and can be written as

$$B(X, Y) = \sum_{i=1}^{n-2} \langle B_i(X), Y \rangle N_i \quad X, Y \in T_p F$$

where the symmetric linear operators $B_i : T_p F \rightarrow T_p F$ are given by $B_i(X) = -(\tilde{\nabla}_X N_i)^T$. The mean curvature field is given by

$$H = \sum_{i=1}^{n-2} \text{Tr}(B_i) N_i.$$

If k_a is the sectional curvature of M along the tangent plane $T_p F$ then, by the Gauss-Codazzi formula, the sectional curvature of F at p is given by

$$k = k_a + \sum_{i=1}^{n-2} \text{Det}(B_i).$$

Fix i for now. Let κ_1 and κ_2 denote the eigenvalues of B_i . Since

$$|H|^2 = \sum_{i=1}^{n-2} \text{Tr}(B_i)^2,$$

we have $|\kappa_1 + \kappa_2| \leq |H| \leq H_0$.

Hence

$$\text{Det}(B_i) = \kappa_1 \kappa_2 = \frac{(\kappa_1 + \kappa_2)^2 - (\kappa_1 - \kappa_2)^2}{4} \leq \frac{H_0^2}{4}.$$

Since M is compact, there is an upper bound K_M on k_a . It follows that the sectional curvature of F is bounded above by $K_0 = K_M + (n-2)H_0^2/4$. \square

Lemma (see Lemma 2.2). *Let (F, g) be a complete Riemannian 2-manifold with sectional curvature bounded above by K_0 . Then the conjugate radius at any $p \in F$ is at least $R := \frac{\pi}{3\sqrt{K_0}}$. Moreover, if we write*

$$\exp^*(g) = dr^2 + f^2(r, \theta) d\theta^2$$

for polar coordinates (r, θ) on $T_p F$ and $r < R$, then $f(r, \theta)$ increasing as a function of r and $f(r, \theta) > r/2$ for all θ .

Proof. Fix polar coordinates (r, θ) on $T_p F$. \exp will denote \exp_p . We know that

$$\exp_* \left(\frac{\partial}{\partial r} \right) = \frac{\partial}{\partial r}.$$

Let

$$J_\theta(r) = \exp_*|_{(r, \theta)} \left(\frac{\partial}{\partial \theta} \right), \quad f(r, \theta) = \|J_\theta(r)\|.$$

Note that

$$\exp^*(g) = dr^2 + f^2(r, \theta) d\theta^2.$$

Fix $\theta > 0$ and regard f as a function r alone. Let $T > 0$ be the smallest value of T such that $J_\theta(T) = 0$. Then f is smooth on $[0, T)$. Assume that $r \in (0, T]$. Since J is a Jacobi field, we have

$$K(r, \theta) f(r, \theta) = -\frac{\partial^2 f(r, \theta)}{\partial r^2},$$

where K denotes the Gaussian curvature of F . Therefore

$$f'' + K_0 f \geq 0.$$

The above inequality combined with

$$f(0) = 0, \quad f'(0) = 1 \quad \text{and} \quad f \geq 0$$

implies that

$$(5) \quad \sin(\sqrt{K_0}r)f'(r) - \sqrt{K_0}\cos(\sqrt{K_0}r)f(r) \geq 0$$

for

$$r \in I = \left[0, \frac{\pi}{2\sqrt{K_0}}\right].$$

Hence

$$f'(r) \geq \sqrt{K_0}\cot(\sqrt{K_0}r)f(r) \geq 0$$

on I . This implies that $f > 0$ on I , since f is non-decreasing on I and $f(r) = 0$ if and only if $f' = 0$ on $(0, r)$, which would contradict $f'(0) = 1$.

Integrating (5), we get

$$f(r) \geq \frac{f(t)}{\sin(\sqrt{K_0}t)}\sin(\sqrt{K_0}r)$$

for $0 < t < r < \frac{\pi}{2\sqrt{K_0}}$.

Letting $t \rightarrow 0$,

$$f(r) \geq \frac{1}{\sqrt{K_0}}\sin(\sqrt{K_0}r) > \frac{r}{2}$$

for $0 < r < \frac{\pi}{3\sqrt{K_0}}$.

Hence we can take $R = \frac{\pi}{3\sqrt{K_0}}$.

□

12. APPENDIX B: LIFTING DISCS UNDER THE EXPONENTIAL MAP

In this section, we prove Lemma 2.3 which allows us to lift discs to the tangent space.

Lemma (see Lemma 2.3). *Let $\iota : B \rightarrow (F, g)$ be an immersion of a disc into a complete Riemannian 2-manifold (F, g) with sectional curvature bounded above by K_0 . Suppose that for the pullback metric ι^*g , the length of $\gamma = \partial B$ and the distance of a point in B to γ are both bounded above by $\epsilon < R/10$ where $R = \frac{\pi}{3\sqrt{K_0}}$. Then for $x = \iota(y)$ in the image of B , there is a lift $\tilde{\iota}$ of ι to the tangent space $T_x F$ so that $\iota = \exp_x \circ \tilde{\iota}$. Furthermore, the lift can be chosen so that $\tilde{\iota}(y)$ is the origin.*

Proof. Recall (Lemma 2.2) that \exp is an immersion on a ball of uniform radius R in $T_p F$ for any $p \in F$. Hence $\exp_x^*(g_j)$ is a metric on this ball and the injectivity radius at 0 is at least R . The tangent space $T_x F$ is a vector space with an inner product with origin identified with x . We seek lifts with respect to the exponential map $\exp_x : T_x F \rightarrow F$.

By the inverse function theorem, Lemma 2.2 yields the following.

Lemma 12.1. *There is a constant $\delta > 0$ such that given any point $\xi = \exp_x(z)$, with $z \in B(0, 9\epsilon) \subset T_x F$, there is a map $\exp_z^{-1} : B(\xi, \delta) \rightarrow T_x F$ with $\exp_x \circ \exp_z^{-1}$ the identity map and $\exp_z^{-1}(\xi) = z$.*

Thus, the exponential map is invertible on sets of diameter less than δ containing a point in the image of $B(0, 9\epsilon)$. Note that δ is not universal. However, none of the constants in the Schwarz lemma depend on δ . Observe that as the metric on the domain B of i is the pullback metric, a set of diameter at most δ in B has image of diameter at most δ .

We shall construct a lift on the disc B by inductively lifting sets b_i of small diameter, as in the proof of the homotopy lifting theorem in Algebraic Topology. However, we need to ensure that at each stage the lift remains within $B(0, 9\epsilon)$ to continue the process. In our situation we can indeed choose such sets b_i using a geometric argument making use of the fact that $\partial B(\gamma)$ has length less than ϵ .

Lemma 12.2. *There is a sequence of smooth balls b_i , each of which has diameter at most ϵ , so that if $B_j = \cup_{i=1}^j b_i$ and B_0 is a single point $B_0 = \{y_0\}$,*

- (1) *The set $B_j \cap b_{j+1}$ is connected and non-empty for $j \geq 0$.*
- (2) *The set $B_j \cap \partial B$ is connected and non-empty for each j .*
- (3) *For each point $p \in B_j$, there is a path α contained in B_j of length at most 2ϵ joining p to $B_j \cap \partial B$.*

Proof. We shall first construct discs b_i with diameter at most δ and then re-order them to satisfy the condition of the lemma.

Consider the function $f : B \rightarrow \mathbb{R}$ given by the distance to the boundary. This is positive on the interior of B and vanishes on the boundary. After a small perturbation, we can assume that this is Morse. Clearly the function f has no local minima in the interior of B .

Thus, f has finitely many critical points of index one and of index 2. As there are no local minima, the descending manifold of each critical points of index 1 is a pair of arcs joining the critical point to the boundary. These partition the disc B into closed subdiscs which we call *basins*. Each of these basins P is the closure of the descending submanifold (i.e., the basin of repulsion of a critical point of index 2 which we denote $O(P)$). We regard the gradient lines from the local maximum in a subdisc as radial lines (see figure ??).

Consider now one such basin P . This is a polygon with $2k$ sides for some $k \geq 0$, with alternate sides contained in the boundary of $B(\gamma)$ and alternate sides consisting of an index one critical point and the descending submanifolds of these.

Consider a closed interval J in $P \cap \partial B$. We define the cone $C(J)$ to be the closure of the set of gradient lines that end in J . If J is in the interior of $P \cap \partial B$ and has boundary points a and b , then $C(J)$ is the region enclosed by J together with the gradient lines joining $O(P)$ to a and b (see figure ??).

Suppose next that one endpoint of J is a vertex v of P and the other is an interior point a . The vertex v is the limit of the gradient line joining an index-one critical point x to V . The cone $C(J)$ is then the region enclosed by J , the gradient line from $O(P)$ to a , and an arc consisting of the closure of the gradient line from $O(P)$ to x and the gradient line from x to v .

In both these cases, we can identify the cone with a sector in the circle, with gradient lines identified with radial lines. Using such an identification, the cone is foliated by lines transversal to the radial lines, namely those corresponding to lines of a fixed distance from the vertex of the sector, which we call *longitudinal arcs* λ_r . We call the point identified with the centre of the circle the *centre* of the cone and the arc J the *boundary arc*.

By bounding the length of J from above, we can ensure that the length of each arc λ_r is less than $\delta/2$. We subdivide the boundary ∂B into closed arcs J_k such that each arc J_k is contained in some basin P_{i_k} with at most one endpoint a vertex, and with the lengths of the arcs J_k sufficiently small to ensure that the corresponding longitudinal arcs in the cones have lengths at most $\delta/2$. We get a partition of B into corresponding subsets $C_k = C(J_k)$.

We can further partition C_k into regions between pairs of longitudinal arcs. For each C_k we choose a collection of longitudinal arcs such that each of the regions between pairs of longitudinal arcs has diameter at most δ . We shall call these regions *squares* (even though the region containing the centre is really a triangle).

This gives a partition of B into discs δ . After re-ordering, these discs will be the regions b_i . Observe that the regions in C_k are naturally ordered starting with the region containing the centre and ending with the region containing the opposite arc. We shall use this as well as the opposite order. We shall often specify whether the first square is the one containing the centre or the boundary arc and consider the corresponding natural order.

We begin by ordering the arcs J_k . Pick an arc J_1 with both endpoints in the interior of an edge of a basin. Order the arcs cyclically beginning with the edge J_1 . Let P_1 be the basin containing J_1 and let y_0 be a point in J_1 .

We shall now order the squares b_i (see figure ?? showing B_j at various stages). Consider the cone on the arc J_1 and let b_1, b_2, \dots, b_{l_1} be the regions of C_1 in the natural order so that b_1 contains J_1 . By construction, for $i \leq l$, $B_i \cap \partial B = J_1$ and each point in B_i is connected to J_1 by a radial line of length less than ϵ .

Next, let k be such that J_2, \dots, J_k are contained in P and J_{k+1} is not (this includes the case when there are only k arcs J_i). We let b_{l_1+1} be the square in J_2 containing the centre of C_2 and let $b_{l_2+2}, \dots, b_{l_2}$ be the other squares in C_2 in the natural order. Observe that for $l_1 < i \leq l_2$, each point in B_i can be connected to the centre by a radial line of length at most ϵ . The centre can in turn be connected to $J_1 \subset \partial B \cap B_i$ by a radial line of length at most ϵ . It is easy to see that the other claims also hold for the sets b_i and B_i constructed so far.

We now continue this process inductively, choosing b_{l_2+1} to be the square of C_3 containing the centre and then choosing successive regions by the natural order. The same argument verifies the claims for these cases. In this manner, we can order all regions in the cones J_1, \dots, J_k to get b_1, \dots, b_l .

Next consider (if we have not exhausted B) the cone C_{k+1} . We take the next square b_{l+1} to be the square in C_{k+1} that contains J_{k+1} . As before, the regions b_{l+2}, \dots , will be the successive regions in C_{k+1} in the natural order upto the region containing the centre. As for the first basin, for the successive interval $J_{k+2}, \dots, J_{k'}$ in the same basin as J_{k+1} , we take regions in successive intervals ordered starting with the region containing the centre.

The above constructions repeated inductively give an ordering of the regions b_i satisfying all the claims. □

Lemma 12.3. *Given a set B_j as above and two points $p, q \in B_j$, there is a path in B_j of length at most 5ϵ joining p to q .*

Proof. The points p and q can be joined to points p' and q' , respectively, in $B_j \cap \partial B$ by paths of length at most 2ϵ . As $B_j \cap \partial B$ is connected and the length of ∂B is at most ϵ , p' and q' can be joined by a path of length at most ϵ . □

We now complete the proof of Lemma 2.3. We construct inductively lifts \tilde{l}_j on the sets B_j . First note that as y_0 and x are in the set B whose diameter is at most 3ϵ , there is a point $z_0 \in B(0, 3\epsilon) \subset T_x F$ with $\exp_x(z_0) = \iota(y_0)$. We define the map \tilde{l} on B_0 by $\tilde{l}(y_0) = z_0$.

Next, we inductively construct a map \tilde{l}_{j+1} on B_{j+1} extending the given map on B_j . Doing this is equivalent to extending the lift on B_j to the set b_{j+1} . First observe that, for a point $\xi \in B_j \cap b_{j+1}$, there is a path β joining ξ to y_0 of length less than 5ϵ . By considering the path $\tilde{l}_j \circ \beta$, it follows that $z = \tilde{l}_j(\xi)$ is contained in the ball of radius 8ϵ in $T_x F_j$. Hence by Lemma 12.1 we can construct a map $\exp_z^{-1} : \iota(b_{j+1}) \rightarrow B(0, 10\epsilon)$ which is a local inverse for the exponential map and so that $\exp_z^{-1}(\iota(\xi)) = z$. We define \tilde{l}_{j+1} on b_{j+1} as $\exp_z^{-1} \circ \iota$. Note that this agrees with the previous definition on ξ . By the inverse function theorem, for each $y_1 \in B_j \cap b_{j+1}$, the inverse image under the exponential map in $B(0, 10\epsilon)$ of y_1 is a discrete set. As $B_j \cap b_{j+1}$ is connected and both $\exp_z^{-1} \circ \iota$ and \tilde{l}_j give lifts of ι on $B_j \cap b_{j+1}$ of the exponential map that agree at the point ξ , it follows that $\exp_z^{-1} \circ \iota$ agrees with \tilde{l}_j on $B_j \cap b_{j+1}$. It follows that we have a well-defined extension \tilde{l}_{j+1} of \tilde{l}_j . Proceeding inductively we obtain a lift \tilde{l} as claimed.

We can ensure that y lifts to the origin by picking a path α , from y to y_0 of length at most 3ϵ . A simpler variation of the above argument gives a lift of this path to a path $\tilde{\alpha}$ beginning at the origin and ending at some point z_0 . We proceed as before with $\tilde{l}(y_0) = z_0$. □

REFERENCES

1. W. K. Allard *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417-491.
2. M. T. Anderson *Curvature estimates and compactness theorems for minimal surfaces in 3-manifolds*, Ann. Sci. c. Norm. Super. IV. Ser. **18** (1985), 89-105.
3. H.I. Choi, R. Schoen *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math. **81** (1985), 387-394.
4. T. H. Colding, W. P. Minicozzi II *Convergence of embedded minimal surfaces without area bounds in three-manifolds*, C. R. Acad. Sci. Paris Sr. I Math. **327** (1998), no. 8, 765-770.
5. M. Gromov *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307-347.
6. J. Hass, G. P. Scott *The existence of least area surfaces in 3-manifolds*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 87-114.
7. C. Hummel *Gromov's compactness theorem for pseudo-holomorphic curves*, Progress in Mathematics, **151**, Birkhäuser Verlag, Basel, 1997.
8. J. Milnor, *Morse theory*, Annals of Mathematics Studies, No. **51**, Princeton University Press, Princeton, N.J. 1963.
9. S. B. Myers *Connections between differential geometry and topology I: Simply connected surfaces*, Duke Math. J. **1** (1935), no. 3, 376-391.
10. S. B. Myers *Connections between differential geometry and topology II: Closed surfaces*, Duke Math. J. **2** (1936), no. 1, 95-102.
11. B. White *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), no. 2, 243-256.
12. J. G. Wolfson *Gromov's compactness of pseudo-holomorphic curves and symplectic geometry*, J. Differential Geom. **28** (1988), no. 3, 383-405.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560003, INDIA
E-mail address: gadgil@math.iisc.ernet.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560003, INDIA
E-mail address: harish@math.iisc.ernet.in